SIGNAL PARAMETER ESTIMATION AND LOCALIZATION VIA MAXIMUM LIKELIHOOD USING A SENSOR ARRAY IN THE PRESENCE OF LEVY NOISE

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ABSTRACT

In this work we investigate an alternative to the stochastic Gaussian *Maximum Likelihood* (ML) method that deals with sub-Gaussian signals. The proposed system is one where the sources are stochastic and Gaussian, and the transfer medium is varying in a highly impulsive manner, introducing the sub-Gaussian nature at the receiver. Alternatively, the impulsive transformation to the signals can be viewed as part of the source model, creating a multivariate source signal whose components cannot be independent and is of impulsiveness equal to the one of the Cauchy distribution.

The Lévy α-stable distribution, of characteristic exponent 0.5 and index of symmetry 1, is used together with the multivariate Gaussian density to model the signal, and the resulting probability density function is derived. Based on this density, the stochastic ML estimator is formulated. A separable solution of the estimator is given, and simulations demonstrating the performance gains relative to the Gaussian-based ML estimator are provided.

1. INTRODUCTION

The Gaussian distribution has traditionally been the most widely accepted distribution and used, as a rule, as a realistic model for various kinds of noise. In recent years however, there has been a tremendous interest in the class of α-stable distributions, which are a generalization of the Gaussian distribution, but are able to model a wider range of phenomena and can be of a more impulsive nature. In fact, the Gaussian is the least impulsive α-stable distribution, while other widely known distributions of this class are the Cauchy and the Lévy.

In 1991, Cambanis, Samorodnitsky, and Taqqu [1] gave a review of α-stable processes from a statistical point of view. Several other statisticians have provided valuable work in the theory of α-stable distributions. In 1993, Nikias and Shao gave an introductory review of α-stable distributions from a statistical signal processing viewpoint followed by a book from the same authors in 1995 [2].

Alpha-stable distributions have been used to model diverse phenomena such as random fluctuations of gravitational fields, economic market indices, and radar clutter. These authors have presented in previous work [3] the appropriateness of the α-stable distributions for modeling noises encountered in audio environments and presented a time delay estimation method for localization of speech sources. Tsakalides and Nikias [4, 5] gave *Maximum Likelihood* (ML) and *Multiple Signal Classification* (MUSIC) based localization algorithms for uncorrelated, impulsive signals. We present in this paper a ML algorithm for signals that are dependent and impulsive in nature.

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1.1. Sub-Gaussian Random Variables

Sub-Gaussian distributions are a special case of α-stable random processes. A Sub-Gaussian random vector X can be defined as a random vector with characteristic function of the form

$$\varphi(u) = \exp\left(\;\; \frac{1}{2}\left[u^T \mathbf{R}\, u\right]^{\alpha/2}\right) \tag{1}$$

where $\mathbf{R}$ is a positive-definite matrix, and the characteristic exponent satisfies $1 < \alpha \leq 2$. Sub-Gaussian processes are variance mixtures of Gaussian processes [6].

If $X(t)$ is sub-Gaussian with parameter α (will be denoted by α-SG($\mathbf{R}$)) and S is a positive stable process with characteristic exponent $\alpha/2$ (*i.e.*, S is $\frac{\alpha}{2}$-stable random variable completely skewed to the right) and $Y(t)$ is a multivariate Gaussian process independent of S, then:

$$X(t) = S^{1/2} Y(t) \tag{2}$$

More information on sub-Gaussian random processes can be found in [1, 2, 7, 8].

2. PROBLEM FORMULATION

The transmitted signals in this case are assumed to be stochastic, and as such, the parameters of interest will be their statistics and *Directions-of-Arrival* (DOA's). Despite the wide variety of optimization criteria used for parameter estimation, the optimal detector is characterized by a single result: the Maximum Likelihood ratio test, which was also one of the first methods to be applied in the area of array signal processing [9]. In this paper we deal exclusively with *Stochastic ML* estimation, where the signals are assumed to be of random rather than of deterministic nature.

We assume a scenario under which there are κ sources received by an array of ρ sensors. The transfer function each signal undergoes while traveling to the array can be modeled as an attenuation and a delay. The attenuation will be considered the same at all sensors under the assumption that the sources are in the far field of the array. These transfer functions are

$$a_{r,k} = \mathrm{e}^{\;\;i\omega\tau_{r,k}}, \quad r = 1\ldots\rho \quad \text{and} \quad k = 1\ldots\kappa \tag{3}$$

where $\tau_{r,k}$ is the delay of the signal (of source k) received at sensor r relative to the first sensor.

We assume the sources to be in the far-field and hence, $\tau_{r,k} = \tau_r(\theta_k)$, and if we are dealing with a linear array, $\tau_{r,k} = (r \quad 1) \cdot \tau_1(\theta_k)$.

We denote the vector of the medium transformations for source k by $a_k = [a_{1,k} \quad a_{2,k} \quad \ldots \quad a_{\rho,k} \quad]^T$.

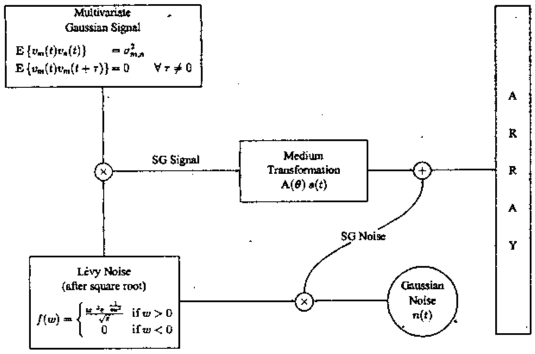

Fig. 1. A multivariate Gaussian signal, corrupted by multiplicative Lévy noise, is then transformed through a set of delays to the receiving end of the array. The additive noise can be generated from the same Lévy process to make it jointly sub-Gaussian with the signal.

The array's input at a single sensor r is

$$x_r(f) = \sum_{k=1}^{\kappa} a_{r,k} \cdot s_k(f) + n_r(f) \tag{4}$$

and therefore, the array's input vector is

$$\boldsymbol{x}(f) = \mathbf{A} \cdot \boldsymbol{s}(f) + \boldsymbol{n}(f) \tag{5}$$

3. SIGNAL MODEL FOR ML ESTIMATION

An alternative to modeling the signal as Cauchy distributed, which was pursued by [4], is using a Sub-Gaussian signal of equal impulsiveness. For this purpose we can employ a distribution of impulsiveness $\alpha = 0.5$, which is completely skewed to the positive axis together with a multivariate Gaussian density. The Lévy distribution satisfies exactly these properties (also referred to as a Paretto type 5 distribution with an index of symmetry $\beta = 1$ and characteristic exponent $\alpha = 0.5$). Fig. 1 gives a top level description of the problem, source, and noise signals:

The Gaussian density is:

$$f(\mathbf{V}) = \prod_{f=f_1}^{f_M} \frac{1}{\pi^{\rho}|\mathbf{R}|} \exp\left(\ \boldsymbol{v}^{\dagger}(f)\mathbf{R}^{\ 1}\boldsymbol{v}(f)\right) \tag{6}$$

where

$$\mathbf{V} = \boldsymbol{v}(f_1), \boldsymbol{v}(f_2), \ldots, \boldsymbol{v}(f_M) \tag{7}$$

and the Lévy distribution shown on Fig. 2 [10] is given by:

$$f(u) = \begin{cases} \frac{u^{\ \frac{3}{2}} e^{\ \frac{1}{4u}}}{2\sqrt{\pi}} & \text{if } u > 0 \\ 0 & \text{if } u < 0 \end{cases} \tag{8}$$

So from eq. (2) the signal $\boldsymbol{s} = [s_1 \ldots s_{\kappa}]^{\mathrm{T}}$ is of the form

$$s_k(t) = u_k(t)^{\frac{1}{2}} \cdot v_k(t) = w_k(t) \cdot v_k(t) \tag{9}$$

It can be shown [11] that the distribution can be given by:

$$f(\boldsymbol{s}) = \frac{1}{2\sqrt{\pi}\,\pi^{\kappa}\,|\boldsymbol{\Sigma}|} \cdot \left[{}^{1}\!/_{4} + \boldsymbol{s}^{\dagger}(t)\boldsymbol{\Sigma}^{\ 1}\boldsymbol{s}(t)\right]^{\ 1} \tag{10}$$

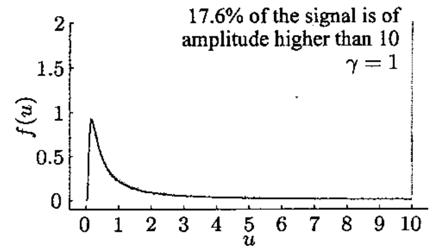

Fig. 2. Lévy Distribution and histogram of the data.

Note that if the Gaussian random variable was one dimensional and real, then under the choice of $\sigma = \sqrt{2}$ the sub-Gaussian random variable would revert to the Cauchy as expected.

$$f(s) = \frac{1}{2\sqrt{2}\pi\sigma} \cdot \left[{}^{1}\!/_{4} + \frac{s^2}{2\sigma^2}\right]^{\ 1}$$

4. MAXIMUM LIKELIHOOD ESTIMATOR

Now the signal $\boldsymbol{x} = [x_1 \ldots x_{\rho}]^{\mathrm{T}}$ is of the form:

$$x_r(t) = y(t)^{1/2} \cdot z_r(t) \tag{11}$$

where, as the transmitted signal, the received signal is sub-Gaussian. It is therefore straightforward to show that the received signal's $\boldsymbol{z}$ statistics will be relating to those of the transmitted signal $\boldsymbol{v}$ by[1]:

$$\mathbf{R} = \mathbf{A}\,\boldsymbol{\Sigma}_v\,\mathbf{A}^{\dagger} + \sigma_n^2 \mathbf{I}_{\rho} \tag{12}$$

Therefore, the maximum likelihood estimator is

$$[\boldsymbol{\Sigma}, \hat{\boldsymbol{\theta}}] = \arg\max_{\hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{\theta}}} \prod_{f=f_1}^{f_M} \frac{1/2}{\sqrt{\pi}\,\pi^{\rho}\,|\mathbf{R}|} \cdot \left[\boldsymbol{x}^{\dagger}(f)\mathbf{R}^{\ 1}\boldsymbol{x}(f) + {}^{1}\!/_{4}\right]^{\ 1} \tag{13}$$

To simplify, take the $\log_e$

$$[\hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{\theta}}] = \arg\min_{\hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{\theta}}} \sum_{f=f_1}^{f_M} \Big\{ \log_e |\mathbf{R}| + \log_e \left[\boldsymbol{x}^{\dagger}(f)\mathbf{R}^{\ 1}\boldsymbol{x}(f) + {}^{1}\!/_{4}\right] \Big\} \tag{14}$$

5. ML – A SEPARABLE SOLUTION

5.1. Estimating the Statistics

We proceed in this case to reach an alternative minimization function to reduce the search space. To do so, we follow the same procedure as in [12] (derivations in [11]) where the ML function is minimized with respect to the signal statistics, assuming known DOA:

$$\boldsymbol{\Sigma}_{\mathrm{ML}} = \sum_{t=t_1}^{t_M} \left[\mathbf{A} \ \left(\frac{\boldsymbol{x}\boldsymbol{x}^{\dagger}}{\mathrm{Tr}\left[\mathbf{R}^{\ 1}\boldsymbol{x}\boldsymbol{x}^{\dagger}\right] + {}^{1}\!/_{4}} \quad \sigma_n^2 \right) \mathbf{A}^{\dagger} \ \right] \tag{15}$$

[1] With the additional assumption that the noise is a sub-Gaussian process produced by the same Lévy sequence, but not necessarily of the same dispersion. A scalar gain is already incorporated in the medium transformation A that can modify the dispersion of the Lévy process.

where $\mathbf{A}^{-} = (\mathbf{A}^{\dagger}\mathbf{A})^{-1}\mathbf{A}^{\dagger}$ and $\mathbf{R}^{-1}$ as defined in eq. 16. The solution of eq. 15 can be easily found using the numerical iteration method.

$$\mathbf{R}^{-1} = \frac{1}{\sigma_n^2}\left\{\mathbf{I} - \mathbf{A}\left(\mathbf{\Sigma}\mathbf{A}^{\dagger}\mathbf{A} + \sigma_n^2\mathbf{I}\right)^{-1}\mathbf{\Sigma}\mathbf{A}^{\dagger}\right\} \qquad (16)$$

An estimate of $\mathbf{R}$ can be used as an initial guess and can be found from the data using a covariation measure. Alternatively, as we have observed, a random or identity initialization matrix also allows for fast convergence.

5.2. DOA Estimation

Clearly, the above assumes that the DOA vector is known, and here we approach the localization part of the problem. Using a pseudo-ML approach we can express the modified ML function irrespective of the statistics $\mathbf{R}$ as

$$\hat{\theta} = \arg\min_{\hat{\theta}} \sum_{f=f_1}^{f_M} \left\{\log_e\left[\boldsymbol{x}^{\dagger}(f)\mathbf{R}^{-1}\boldsymbol{x}(f) + {}^{1}/_{4}\right]\right\} \qquad (17)$$

where $\mathbf{R}$ can be substituted with any valid statistics (identity matrix for instance). A search algorithm, such as the one described in [13], or an alternating line search, can be used to find the solution of the above equation.

6. SIMULATIONS

6.1. DOA Estimation

Several sets of simulations need to be performed to test the validity of the algorithm. In each DOA estimation test, $\mathbf{\Sigma} = \mathbf{I}$ is assumed to hold, although the test matrix has a random correlation structure, but always with diagonal elements of dispersion equal to 1.

In all cases the impulsiveness was kept constant ($\alpha = 1$ for cases 1 & 4, and $\alpha = 2$ for 2 & 3 as described below). The Generalized Signal-to-Noise Ratio used below is defined as:

$$\text{GSNR} = 10\log_{10}\left(\frac{\gamma_s}{\gamma_n}\right) = -10\log_{10}(\gamma_n) \qquad (18)$$

Fig. 3 shows the mean squared error and the probability of localization for the conditions described in Fig. 1. Four cases were simulated, in each one the noise follows the same assumptions as the signal:

1. Exactly as per the derivation assumptions (Fig. 3a): received signal is sub-Gaussian, created from a Multivariate Gaussian and a univariate Lévy (can be viewed as Lévy energy fluctuation). Received signal impulsiveness is $\alpha = 1$.
2. The signal is a Multivariate Gaussian (Fig. 3b), and is created from a Multivariate Gaussian ($\boldsymbol{v}$) and a univariate Gaussian (w). Received signal impulsiveness is $\alpha = 2$.
3. The signal is a Multivariate Gaussian (Fig. 3c) and it undergoes *no* energy fluctuation ($w = 1$, $\boldsymbol{v} = \boldsymbol{s}$). This conforms to the assumptions of the well known Gaussian based ML. Clearly, the received signal impulsiveness is $\alpha = 2$.
4. Finally, the received signal is sub-Gaussian(Fig. 3d), created from a Multivariate Gaussian ($\boldsymbol{v}$) and a *Multivariate* Lévy ($\boldsymbol{w}$). In this case, the signals can be viewed as simply Cauchy or as Gaussian with a different Lévy energy fluctuation for each source. Received signal impulsiveness is $\alpha = 1$.

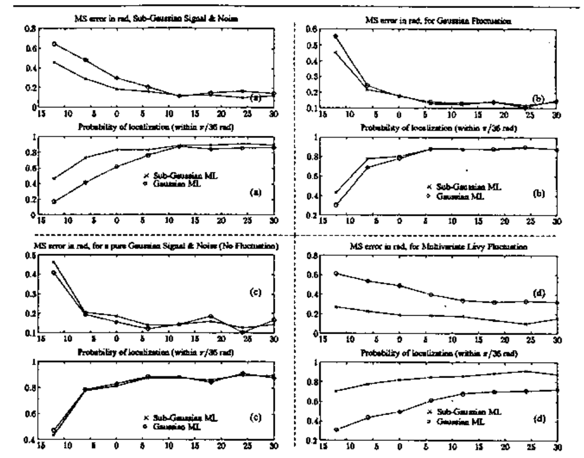

Fig. 3. Localization simulations for different noise conditions as described in text.

As expected, the sub-Gaussian ML method performs better when the derivation assumptions hold (Fig. 3a). Likewise, when the signal is a multivariate Gaussian, the Gaussian ML algorithm performs better (Fig. 3c).

In the cases that neither assumption holds however, we can see how more robust the sub-Gaussian ML method is. When the signal follows (2) (Fig. 3b), the sub-Gaussian ML performs slightly better than the Gaussian ML. The real benefit of the proposed ML method can however be observed when the signals are impulsive due to random multiplicative noise, independent from one source to the next (Fig. 3d).

6.2. Statistics Estimation using ML

Fig. 4 shows the statistic estimates for a 3-source problem with

$$\mathbf{\Sigma} = \begin{bmatrix} 2 & 1-0.4i & 1.0-1.6i \\ 1+0.4i & 4 & 0.3-0.8i \\ 1+1.6i & 0.3+0.8i & 3 \end{bmatrix}$$

when the initialization matrix in eq. (15) was the identity matrix.

The sample statistics are slightly different from the above depending on the length of the realization, and are plotted on Fig. 4 as well. The histogram plots show on the positive side the sample statistics, and on the negative side the estimates of the diagonal elements of $\hat{\Sigma}$ as estimated by eq. (15). The insignificantly small complex components of the diagonal of $\hat{\Sigma}$ are ignored due to prior knowledge. The scatter plots present on an Argand diagram the off-diagonal elements of the statistics matrix. The dots denote the actual sample statistics, while the estimates are shown with '$\times$'.

As can be observed from Fig. 4, the number of sensors is far more important than the total number of samples. As an example, we can see that cases (a) and (b) have the same overall number of samples, but the performance is far superior in case (b) where the number of sensors is 4 times the ones in (a). In fact, a significant decrease of SNR in (c) can be compensated by an increase in the number of sensors. Likewise, we can observe that even a significant increase in SNR from (d) to (e) provides little improvement in the accuracy of the estimates, while an increase in the number of sources improves the accuracy dramatically in (f).

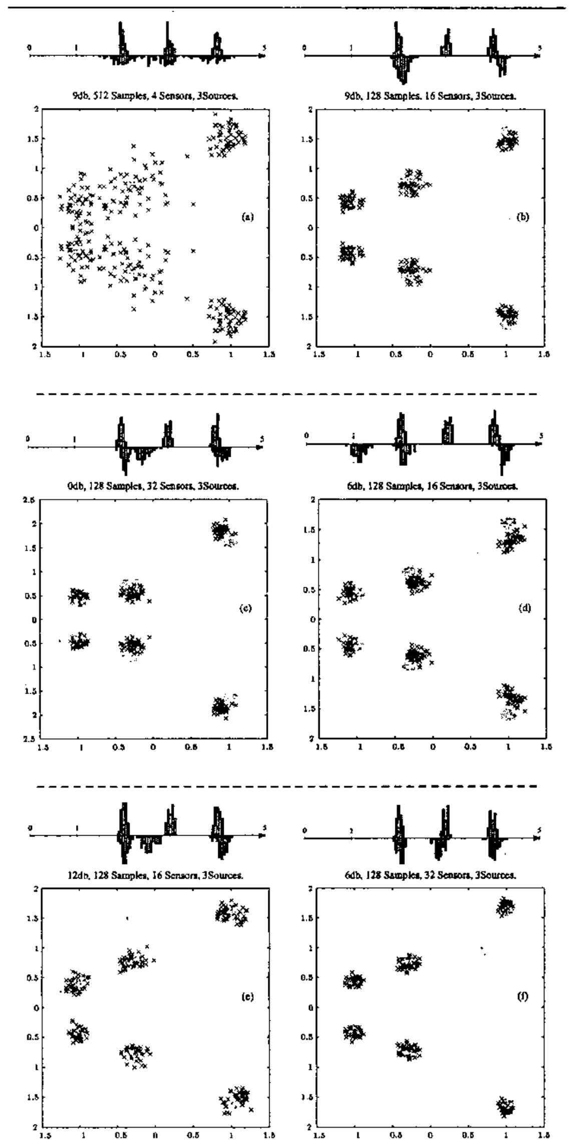

Fig. 4. Simulations show the effectiveness of the separable ML estimation of statistics for a 3-source problem under various noise conditions and array arrangements.

7. CONCLUSIONS

We have presented a ML algorithm that is robust in that it can operate under both impulsive and Gaussian signal conditions. The suggested model can be used in several different scenarios: Gaussian sources undergoing the same energy fluctuations, dependent and impulsive sources, or even independent and impulsive sources. One possible application is multipath signals, which can be highly dependent and impulsive, or even echo in an audio environment.

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